**Ex 1. 7** *Given the following joint distribution on $(X, Y)$ (see table)*

| $X\backslash Y$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| 1 | $\frac{1}{6}$ | $\frac{1}{12}$ | $\frac{1}{12}$ |
| 2 | $\frac{1}{12}$ | $\frac{1}{6}$ | $\frac{1}{12}$ |
| 3 | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{6}$ |

*Let $\widehat{X}(Y)$ be an estimator for $X$ (based on $Y$) and let $P_e = Pr(\widehat{X}(Y) \neq X)$.*

- *Find the minimum probability error estimator $\widehat{X}(Y)$ and the associated $P_e$.*
- *Evaluate Fano inequality for this problem and compare.*

**Ex 1. 9** *Consider the random variable defined by the following table*

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0.49 & 0.26 & 0.12 & 0.04 & 0.04 & 0.03 & 0.02 \end{pmatrix}$$

$x_{45}$ $x_{67}$

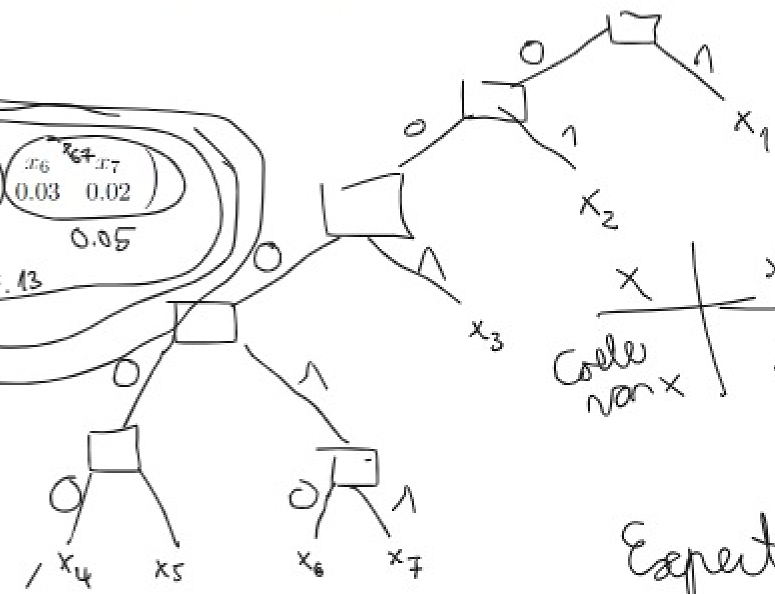
0.08 0.05

0.13

0.25

0.51

- *Find a binary Huffman code for $X$.*
- *Find the expected codelength for this encoding.*
- *Find a ternary Huffman code for $X$.*




Präfixcode

| $X$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ |
|---|---|---|---|---|---|---|---|
| Code von $X$ | 1 | 01 | 001 | 00000 | 00001 | 00010 | 00011 |

Expected codelength $= 0{,}49 \cdot 1$
$+ 0{,}26 \cdot 2 +$
$+ 0{,}12 \cdot 3 +$
$+ 0{,}04 \cdot 5 +$
$+ 0{,}04 \cdot 5 +$
$+ 0.03 \cdot 5$
$+ 0.02 \cdot 5 =$

$= 0.49 \cdot 1 + 0.26 \cdot 2 + 0.12 \cdot 3 + 5 \cdot (0.04 + 0.04 + 0.03 + 0.02) =$

$= \underline{\underline{2{,}02}}$

Dekodieren

$\underbrace{001}_{x_3}1 \xrightarrow{\text{Dekodieren}} x_3 x_1$

$0011 \xrightarrow{\text{Dk}} x_7$

miro

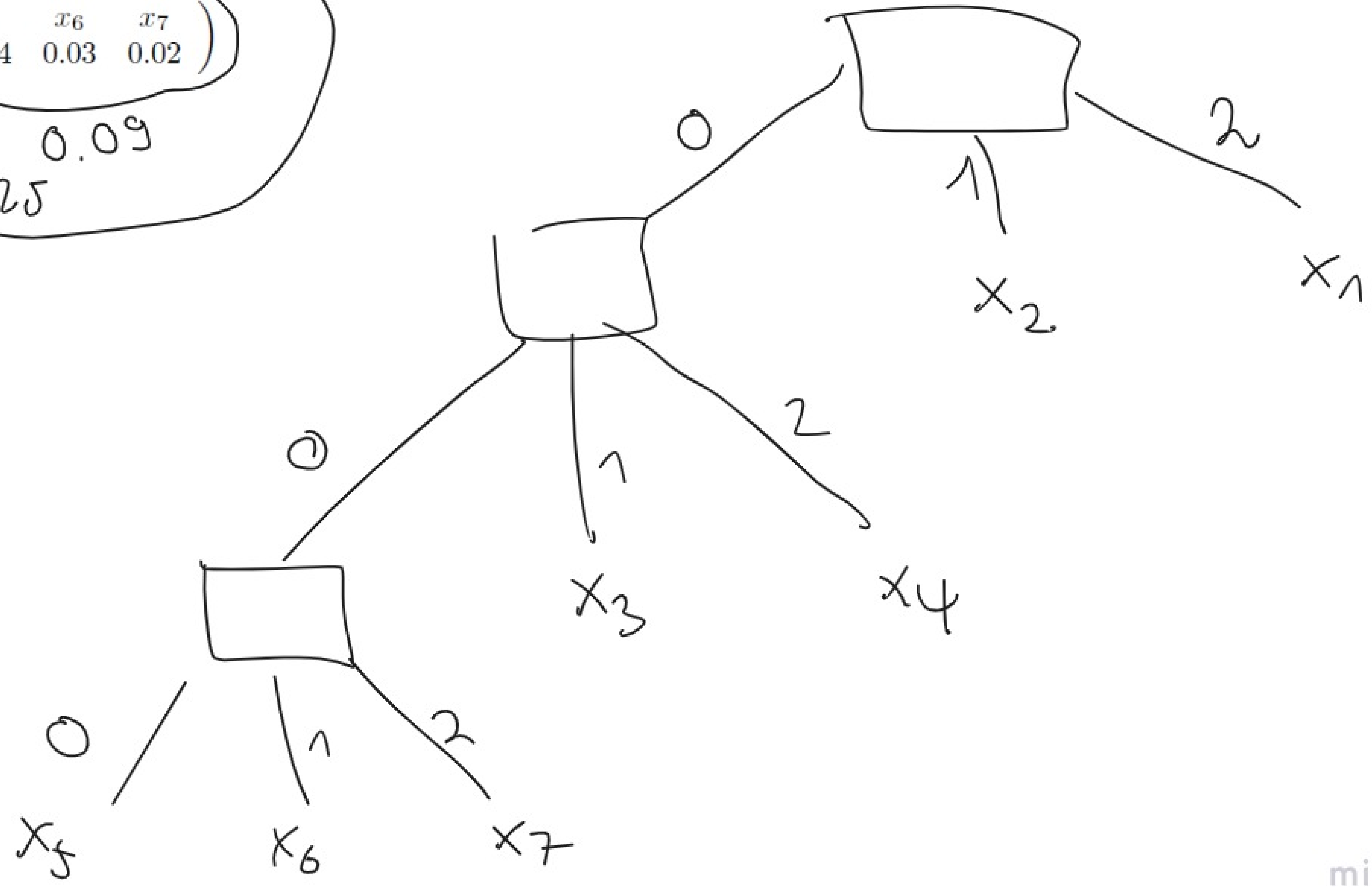
**Ex 1. 9** *Consider the random variable defined by the following table*

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0.49 & 0.26 & 0.12 & 0.04 & 0.04 & 0.03 & 0.02 \end{pmatrix}$$

0.09

0.25

- *Find a binary Huffman code for $X$.*
- *Find the expected codelength for this encoding.*
- *Find a ternary Huffman code for $X$.*



**Ex 1. 7** *Given the following joint distribution on $(X, Y)$ (see table)*

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| 3 | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{6}$ |

*Let $\widehat{X}(Y)$ be an estimator for $X$ (based on $Y$) and let $P_e = Pr(\widehat{X}(Y) \neq X)$.*

- *Find the minimum probability error estimator $\widehat{X}(Y)$ and the associated $P_e$.*
- *Evaluate Fano inequality for this problem and compare.*

Bsp.: $\hat{X}(a) = 1$

$\hat{X}(b) = 2$

$\hat{X}(c) = 3$

$$\hat{X} : \overset{Y}{\overset{\wedge}{\{a, b, c\}}} \to \{1, 2, 3\}$$

$a \mapsto 1$

$b \mapsto 2$

$c \mapsto 3$

$$P_e = Pr(\hat{X}(Y) \neq X) = \mathbb{P}(Y = a, \overset{\overset{1}{\wedge}}{\overset{\shortparallel}{\hat{X}(a)}} \neq X) + \mathbb{P}(Y = b, \overset{2}{\overset{\shortparallel}{\hat{X}(b)}} \neq X) + \mathbb{P}(Y = c, \hat{X}(c) \neq X)$$

$$= \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{6}{12} = \frac{1}{2}$$

**Ex 1. 5** *Let $p(x, y)$ be given as in the table.*

| $X \backslash Y$ | 0 | 1 |
|---|---|---|
| 0 | $\frac{1}{3}$ | $\frac{1}{3}$ |
| 1 | 0 | $\frac{1}{3}$ |

$\mathbb{P}(X=0, Y=0)$

$\mathbb{P}(X=0)$

$\frac{2}{3}$, $\frac{1}{3}$

$\frac{1}{3}$, $\frac{2}{3}$, $1$

$\mathbb{P}(Y=0) = \mathbb{P}(Y=0, X=0) + \mathbb{P}(Y=0, X=1)$

$\mathbb{P}(Y=1)$

*Find*

- *a)* $H(X)$, $H(Y)$
- *b)* $H(X|Y)$, $H(Y|X)$
- *c)* $H(X,Y)$
- *d)* $H(Y) - H(X|Y)$
- *e)* $I(X;Y)$
- *Draw a Venn diagram for the quantities in (a) through (e).*

a) $H(X) = -\sum_{x \in \mathcal{X}} p_X(x) \cdot \log(p_X(x))$, $\mathcal{X} = \{0,1\}$

$$= -\left(p_X(0) \cdot \log(p_X(0)) + p_X(1) \cdot \log(p_X(1))\right)$$

$$= -\left(\frac{2}{3} \cdot \log_2\left(\frac{2}{3}\right) + \frac{1}{3} \cdot \log_2\left(\frac{1}{3}\right)\right)$$

$$= -\left(\frac{2}{3} \cdot (\log_2(2) - \log_2(3)) + \frac{1}{3} \cdot (\log_2(1) - \log_2(3))\right)$$

$$= -\left(\frac{2}{3}(1 - \log_2(3)) + \frac{1}{3}(0 - \log_2(3))\right)$$

$$= -\left(\frac{2}{3} - \log_2(3)\right)$$

$$= \log_2(3) - \frac{2}{3}$$

$$\boxed{\log\left(\frac{a}{b}\right) = \log(a) - \log(b)}$$

$\log_2(2) = 1$, da $2^{\boxed{1}} = 2$

$\log_2(1) = 0$, da $2^0 = 1$

Mädchen, Jungen, 500 Kinder, Brille oder kein

| | M | J | |
|---|---|---|---|
| B | 100 | 50 | 150 |
| $\bar{B}$ | 100 | 250 | 350 |
| | 200 | 300 | 500 |

$$\mathbb{P}(B \mid J) = \frac{50}{300} = \frac{1}{6}$$

$$\mathbb{P}(B \mid M) = \frac{100}{200} = \frac{1}{2}$$

$$\mathbb{P}(M \mid B) = \frac{100}{150}$$

$$\mathbb{P}(J \mid \bar{B}) = \frac{250}{350}$$

$$H(Y) = \log_2(3) - \frac{2}{3}$$

$$H(X|Y) =$$

$$= -\sum_{x,y \in \mathcal{X}\times\mathcal{Y}} p_{X,Y}(x,y) \log(p_{X|Y}(x|y))$$

$$= -p_{X,Y}(0,0) \cdot \log(p_{X|Y}(x=0|y=0))$$

$$- p_{X,Y}(1,1) \log(p_{X|Y}(x=1|y=1))$$

$$- p_{X,Y}(1,0) \log(p_{X|Y}(x=1|y=0))$$

$$- p_{X,Y}(0,1) \log(p_{X|Y}(x=0|y=1)) =$$

**Ex 1. 5** *Let $p(x, y)$ be given as in the table.*

| $X\backslash Y$ | 0 | 1 |
|---|---|---|
| 0 | $\frac{1}{3}$ | $\frac{1}{3}$ |
| 1 | 0 | $\frac{1}{3}$ |

*Find*

- *a)* $H(X)$, $H(Y)$
- *b)* $H(X|Y)$, $H(Y|X)$
- *c)* $H(X, Y)$
- *d)* $H(Y) - H(X|Y)$
- *e)* $I(X; Y)$
- *Draw a Venn diagram for the quantities in (a) through (e).*

$$= -\left(\frac{1}{3}\underbrace{\log(1)}_{=0} + \frac{1}{3}\underbrace{\log\left(\frac{1}{2}\right)}_{=-1} + 0 + \frac{1}{3}\underbrace{\log\left(\frac{1}{2}\right)}_{=-1}\right) =$$

$$2^{\boxed{-1}} = \frac{1}{2}$$

$$= \underline{\underline{\frac{2}{3}}}$$